

Mathematical Support to Braneworld Theory

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Abstract

The braneworld theory appear with the purpose of solving the problem of the hierarchy of the fundamental interactions. The perspectives of the theory emerge as a new physics, for example, deviation of the law of Newton's gravity. One of the principles of the theory is to suppose that the braneworld is local submanifold in a space of high dimension, the bulk, solution of Einstein's equations in high dimension. In this paper we approach the mathematical consistency of this theory with a new proof of the fundamental theorem of submanifolds for case of semi-Riemannian manifolds. This theorem consist an essential mathematical support for this new theory. We find the integrability conditions for the existence of space-time submanifolds in a pseudo-Euclidean space.

1 Brief Historic of Immersions of the Space-times

One of the main problems of Riemannian geometry is the immersion of a Riemannian manifold in Euclidean space. This problem consists to know when a Riemannian manifold M , with metric g , permits an immersion $f : M \longrightarrow R^n$ such that the induced metric has its origin in the scalar product of R^n in the submanifold $f(M) \subset R^n$ and coincides with g . When f exists we say that M is isometrically immersed into R^n .

This problem can be approached under some aspects. We can require the isometric immersion f to be defined on M (global case) or only in the neighborhood of some point of M (local case). In general the dimension of the Euclidean space

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in which a immersion of M is realized is high, which depends of the regularity of the immersion. When is possible to release an isometric immersion for sufficiently large n it another question appears: the least possible value of n . Some other questions appear from immersion problem, for instance: isometric deformation, extrinsic absolute invariants, algebraic criteria for immersion class, rigid immersion, etc. Nowadays there is a great literature about immersion, embedding (imbedding), and submanifolds connected with others branches of mathematics.

The oldest publication cited in contemporaneous articles about isometric immersion is of 1873 by Schlaefli, where he made a conjecture: Any Riemannian manifold m dimensional can be isometrically immersed into Euclidean space $D = \frac{m(m+1)}{2}$ dimensional. In 1926, Janet published a proof of the conjecture, but in 1931 Burstin gave a correct proof. Cartan gave a correct proof too, but used differential forms. There are many results in the literature, some of them are: Nash's theorem (global immersions), O'Neill (algebraic criteria for immersion), Friedman's theorem (generalization of Bustin's theorem for pseudo-Riemannian manifolds and analytical metrics), Greene's theorem (theorem for pseudo-Riemannian manifolds and differential metrics), etc.

The physics of gravitational fields is intimately connected with the geometry of four-dimensional pseudo-Riemannian manifold (space-times) whose metrics obey Einstein's field equations. The Einstein's theory customarily has been treated as an intrinsic geometric theory where the four-dimensional space-time consists of an arena of the physics events perfectly according with pseudo-Riemannian geometry. We live in a four dimensional world. However the idea of unification of fundamental interactions leads us to higher dimensional theories, where now the arena of physical events passes to be a high dimension space and without doubt we must have our world immersed into the latter.

The immersion problem which involves pseudo-Riemannian manifolds is brought to multidimensional physics where it may help in finding new geometrical characterizations of gravitational fields which can be connected to physics. In submanifolds (space-times) this problem is called embedding of space-times.

The first result about embedding of space-times came from the paper of Kasner, 1921, "Finite Representation of the Solar Gravitational Field in Flat Space of Six Dimensions". He gave an explicit local isometric embedding of the external Schwarzschild solution. Others results about this topic have been found in the literature with various physical motivations, or simply for mathematics motivations.

In 1926 Kaluza and Klein tried to explain both gravitational interaction and the electromagnetic interaction in one theory. They worked in a five dimensional space-time, where one of the space dimensions is compact. It was extended to include all the gauge interactions. Thus to accommodate the strong, weak and electromagnetic interactions we need at least seven extra dimensions plus four dimensions of our world. Only in the eighties Kaluza-Klein theory was derived from the hypothesis

that the four-dimensional space-time is locally and isometrically embedded in higher dimensional space.

In 1965 occurred a great event with mathematicians and physicists interested on the embedding problem motivated in properties of particle physics in curved space-time, identifying the internal symmetries of particle physics with the symmetries of the normal piece of embedding space.

The embedding problem has been required in problems linked to minimal class of the embedding, extrinsic gravity, theory of strings and membranes. Nowadays the embedding problem emerged with a new theory - braneworld - a theory that lead us the idea of unification of fundamental interactions using extra dimensions. Such a model has been showing positive in the sense that we find perspectives and probably deep modifications in the physics, such as: unification in a TeV scale, quantum gravity in this scale and deviation of Newton's law of gravity for small distances.

A braneworld may be regarded as a space-time locally embedded in a higher dimensional space, the bulk, solution of higher dimensional Einstein's equations. Furthermore, the embedded geometry is assumed to exhibit quantum fluctuations with respect to the extra dimensions at the TeV scale of energies. Finally, all gauge interactions belonging to the standard model must remain confined to the four-dimensional space-time. Contrasting with other higher dimensional theories, the extra dimensions may be large and even infinite, with the possibility of being observed by TeV accelerators. The embedding conditions relate the bulk geometry to the braneworld geometry, as it is clear from the Gauss-Codazzi-Ricci equations. [1]

The fundamental theorem of Riemannian geometry warrants that a given Riemannian manifold can be locally and isometrically immersed in higher dimensional Euclidean spaces, if and only if there exist tensors that satisfy the Gauss-Codazzi-Ricci equations. [2, 3, 4]

However we give a new proof of a fundamental and classical theorem for manifolds with arbitrary signatures and dimensions. This theorem is very interesting when we consider a space-time of the General Relativity as a submanifold of a pseudo-Euclidean space.

Braneworlds may be described as a family of stable perturbations of a given locally embedded background space-time. This suggests having a dynamic bulk whose geometry depends on that of the braneworld. From the point of view of perturbations in braneworlds we can suppose that there is a fundamental state (ground state) of the bulks named a induced pseudo-Euclidean bulk so that any braneworld would be immersed in some appropriate flat space bulk. This immersion had been guaranteed by the fundamental theorem of submanifolds.[8, 9]

2 Preliminaries

Consider a point $p \in (V_n, g)$, where (V_n, g) is a pseudo Riemannian manifold (braneworld) with $r - s$ signature, given by local coordinates $x^i, i = 1, \dots, n$. We have that the local and isometric immersion $Y : (V_n, g) \rightarrow (V_D, \eta)$, where (V_D, η) is a pseudo Euclidean manifold (bulk) with $p - q$ signature, the position of $p \in V_n$ is indicated for $Y^\mu = Y^\mu(x^1, \dots, x^n)$. In local coordinates the induced metric is given by

$$g_{ij} = \eta_{\mu\nu} Y_{,i}^\mu Y_{,j}^\nu, \quad (1)$$

where $Y_{,i}^\mu = \frac{\partial Y^\mu}{\partial x^i}$; the local orthonormal basis of $T_p V_n$ is $\{\frac{\partial}{\partial x^i}\}$ and another local orthonormal basis of $T_{Y(p)} V_D$ is $\{\frac{\partial}{\partial Y^\mu}\}$.

Set ∇' the Levi-Civita connection of V_D , where U' and $W' \in V_D$. We decompose $\nabla'_{U'} W'$ on V_D , when we restrict the differential fields $U'|_U$ and $W'|_W$, on the fiber TV_n . The decomposition of $\nabla'_U W$ is given by

$$\nabla'_U W = \nabla_U W + b(U, W), \quad (2)$$

where the tangent part $\nabla_U W$ is the projection of $\nabla'_U W$ on the TV_n . The normal part $b(U, W)$ is defined by eq. (2), it is a 2-linear symmetric application.

$$b : TV_n \times TV_n \rightarrow T(V_n)^\perp,$$

we called it the second fundamental form of V_n . Observe that $\nabla_U W$ is a Levi-Civita connection of V_n , agreeing with the Levi-Civita connection of the induced metric. We have that $T(V_n)^\perp$ is the orthogonal complement of TV_n in $TV_D|_{V_n}$. [5]

There exists also an induced connection on the fiber $T(V_n)^\perp$. This connection is defined by the application,

$$\nabla'' : TV_n \times T(V_n)^\perp \rightarrow T(V_n)^\perp$$

with

$$\nabla'_U N = \nabla''_U N + S(U, N) \quad (3)$$

the decomposition of $\nabla'_U N$, where the tangent part is given by the linear application,

$$S : TV_n \times T(V_n)^\perp \rightarrow TV_n$$

related with b through

$$\eta(S(U, N), N) = \eta(b(U, W), N) \quad (4)$$

while $\nabla''_U N$ is the normal part of $\nabla'_U N$ in eq. (3). [5]

In local coordinates we can write,

$$U = u^i \frac{\partial}{\partial x^i}, \quad W = w^j \frac{\partial}{\partial x^j}, \quad N_A = N_A^\mu \left(\frac{\partial}{\partial Y^\mu} \right)^\perp,$$

where $\left\{ \frac{\partial}{\partial Y^\mu} \right\}^\perp$, $\mu = D - n, \dots, D$ is local and orthonormal basis of $T_p(V_n)^\perp$, with

$$\eta(N_A, N_B) = \eta_{AB} = \epsilon_A g_{AB} \quad (\text{com } \epsilon_A = \pm 1) \quad (5)$$

and

$$\eta(N_A, U) = 0. \quad (6)$$

We indicate N_A any normal vector. We have that,

$$\nabla_U W = u^i (w^k_{,i} + \Gamma_{ij}^k w^j) \frac{\partial}{\partial x^k}, \quad (7)$$

$$\nabla_U N_A = u^i (N_{A,i}^\mu + A_{i\nu}^\mu N_A^\nu) \left(\frac{\partial}{\partial Y^\mu} \right)^\perp. \quad (8)$$

where $\mu, A, B, \dots = D - n, \dots, D$. Thus, $A_{i\mu\nu} \equiv A_{iAB}$ and $b\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) = b_{ij}{}^A \left(\frac{\partial}{\partial Y^A}\right)^\perp$. The second form coefficients b_{ijA} are the components of $D - n$ tensors on $T_p V_n$, for each fixed A , with $b_{ijA} = b_{jiA}$. On the other hand, $A_{iAB} = -A_{iBA}$ are components of $\binom{D-n}{2}$ vectors on $T_p V_n$, it corresponds to the normal part of the connection on V_D and we called it twisting vector.[6]

Replacing $U = \frac{\partial}{\partial x^i}$, and $N_A = \left(\frac{\partial}{\partial Y^A}\right)^\perp$ in the equations (2) and (3) we obtain,

$$Y_{;ij}^\mu = g^{AB} b_{ijA} N_B^\mu \quad (9)$$

$$N_{A,j}^\mu = -g^{mn} b_{jmA} Y_{,n}^\mu + g^{MN} A_{jAM} N_N^\mu. \quad (10)$$

where $Y_{;i}^\mu$ is the covariant derivate of Y^μ with respect the g_{ij} metric. The equations (9) and (10) are called: Gauss formula and Weingarten formula, respectively.

It is easy to see from eq. (9) that the second form coefficients are given by

$$b_{ijA} = Y_{;ij}^\mu N_A^\nu \eta_{\mu\nu}. \quad (11)$$

From eq. (10) we obtain the twisting vector components

$$A_{iAB} = N_{A,B,i}^\nu \eta_{\mu\nu}. \quad (12)$$

We have that the system of partial differential equations from eqs. (9) and (10) can be resolved for arbitrary initial vectors of $\{Y_{,i}^\mu, N_A^\mu\}$ for a fixed point, so that they satisfy,

$$g_{ij} = Y_{,i}^\mu Y_{,j}^\nu \eta_{\mu\nu}, \quad (13)$$

$$N_A^\mu Y_{,i}^\nu \eta_{\mu\nu} = 0, \quad (14)$$

$$N_A^\mu N_B^\nu \eta_{\mu\nu} = g_{AB} = \epsilon_A \delta_{AB}, \quad (15)$$

where $\epsilon_A = \pm 1$ and depends from the (V_D, η) signature.

In the next section we are going to prove the fundamental theorem of the submanifolds. Particularly in the case of braneworlds (space-times) submanifolds the theorem appear as mathematical support to braneworld theory.

3 The Fundamental Theorem of the Submanifolds

A pseudo Riemannian manifold (V_n, g) with $r - s$ signature, is locally and isometrically immersed in the pseudo Euclidean manifold (V_D, η) with $p - q$ signature, if only if there exist $D - n$ symmetric matrices b_{ijA} and $\binom{D-n}{2}$ vectors A_{iAB} satisfying the Gauss-Codazzi-Ricci equations.

Proof:

Suppose that there exists a local and isometric immersion

$$\begin{aligned} Y : (V_n, g) &\rightarrow (V_D, \eta) \\ x^i &\rightarrow Y^\mu = Y^\mu(x^i) \end{aligned}$$

If to a fixed point we choose arbitrary tangent vectors $Y_{,i}^\mu$ and normal vectors N_A^μ , then these vectors satisfy eqs. (13), (14) and (15) from section (2). On the other hand, the set $\{Y_{,i}^\mu, N_A^\mu\}$ needs to satisfy the Gauss and Weingarten formulas given by the system

$$Y_{,ij}^\mu = g^{AB} b_{ijA} N_B^\mu - \Gamma_{ij}^r Y_{,r}^\mu \quad (16)$$

$$N_{A,j}^\mu = -g^{ml} b_{jmA} Y_{,l}^\mu + g^{MN} A_{jAM} N_N^\mu, \quad (17)$$

where $b_{ijA} = Y_{,ij}^\mu N_A^\nu \eta_{\mu\nu}$ and $A_{iAB} = N_A^\mu N_B^\nu \eta_{\mu\nu}$, warranted the existence of these tensors. This system is super determined and since Y^μ already exists the system (1) and (2) has solution. Therefore the integrability conditions

$$Y_{,ijk}^\mu = Y_{,ikj}^\mu \quad (18)$$

$$N_{A,jk}^\mu = N_{A,kj}^\mu \quad (19)$$

are satisfied for $D - n$ matrices (b_{ijA}) and $\binom{D-n}{2}$ A_{iAB} vectors. These integrability conditions, consequence of Frobenius theorem, conduce to the Gauss-Codazzi-Ricci equations.[7]

Using (1) calculate (3) and use the Weingarten formula for $N_{B,k}^\mu$ and $N_{B,j}^\mu$. We obtain,

$$\begin{cases} g^{AB}b_{ijA,k}N_B^\mu + g^{AB}b_{ijA}(-g^{m\ell}b_{kmB}Y_{,\ell}^\mu + g^{MN}A_{kBM}N_N^\mu) \\ -\Gamma_{ij,k}^r Y_{,r}^\mu - \Gamma_{ij}^r Y_{,rk}^\mu + g_{,k}^{AB}b_{ijA}N_B^\mu = \\ g^{AB}b_{ikA,j}N_B^\mu + g^{AB}b_{ikA}(-g^{m\ell}b_{jmB}Y_{,l}^\mu + g^{MN}A_{jBM}N_N^\mu) \\ -\Gamma_{ik}^r Y_{,rj}^\mu + g_{,j}^{AB}b_{ikA}N_B^\mu \end{cases} \quad (20)$$

Using that $\{x^i\}$ are geodesic coordinates, then $\Gamma_{ij}^k = 0$. Multiply (5) by $\eta_{\mu\nu}Y_{,n}^\nu$ and employ the eqs. (13), (14) and (15) from section (2) to obtain the Gauss equation,

$$R_{nijk} = g^{AB}(b_{ikA}b_{jnB} - b_{ijA}b_{knB}) \quad (21)$$

Again using that $\{x^i\}$ are geodesic coordinates, multiplying (5) by $\eta_{\mu\nu}N_C^\nu$ and employing again eqs. (13), (14) and (15) from section (2) we obtain,

$$b_{ijC,k} - b_{ikC,j} = g^{AB}(b_{ikA}A_{jBC} - b_{ijA}A_{kBC}). \quad (22)$$

We have a geodesic system, then we can write the Codazzi equation,

$$b_{ijC;k} - b_{ikC;j} = g^{AB}(b_{ikA}A_{jBC} - b_{ijA}A_{kBC}). \quad (23)$$

Now take (2), calculate (4) and use the Weingarten formula for $N_{N,k}^\mu$ and $N_{N,j}^\mu$. We obtain,

$$\begin{aligned} & -g^{m\ell,k}b_{jmA}Y_{,\ell}^\mu - g^{m\ell}b_{jmA,k}Y_{,\ell}^\mu - g^{m\ell}b_{jmA}Y_{,\ell k}^\mu + g_{,k}^{MN}A_{jAM}N_N^\mu \\ & + g^{MN}A_{jAM,k}N_N^\mu + g^{MN}A_{jAM}(-g^{m\ell}b_{kmN}Y_{,\ell}^\mu + g^{CD}A_{kNC}N_D^\nu) = \\ & -g_{,j}^{m\ell}b_{kmA}Y_{,\ell}^\mu - g^{m\ell}b_{kmA,j}Y_{,\ell}^\mu - g^{m\ell}b_{kmA}Y_{,\ell j}^\mu - g_j^{MN}A_{kAM}N_N^\mu \\ & + g^{MN}A_{kAM,j}N_N^\mu + g^{MN}A_{kAM}(-g^{m\ell}b_{jmN}Y_{,\ell}^\mu + g^{CD}A_{jNC}N_D^\mu) \end{aligned} \quad (24)$$

multiplying (9) by $\eta_{\mu\nu}N_B^\nu$, employing again eqs. (13), (14) and (15) from section (2) and using the definition of the second form we obtain the Ricci equation,

$$\begin{aligned} & A_{jAB;k} - A_{kAB;j} = \\ & g^{MN}(A_{jAM}A_{kNB} - A_{kAM}A_{jNB}) + g^{ml}(b_{kmA}b_{ljB} - b_{jmA}b_{lkB}) \end{aligned} \quad (25)$$

We conclude that if there exists an immersion, then there exist $(D-n)$ symmetric b_{ijA} and $\binom{D-n}{2} A_{iAB}$ vectors satisfying the Gauss-Codazzi-Ricci equations.

Now we prove the sufficiency condition. We know that if for a fixed point we choose arbitrary vectors $Y_{,i}^\mu$ and N_A^μ satisfying eqs (13), (14) and (15) section (2),

then Y^μ is determined. Therefore we need to calculate $\{Y_{,i}^\mu N_A^\mu\}$. For this realization consider the system equations of (1) and (2), where we know to exist by hypothesis $(D-n)$ (b_{ijA}) and $(1/2)(D-n)(D-n-1)$ A_{iAB} . We observe that the system (1) and (2) is a system of partial differential equations super determined. The Frobenius theorem warrants the integrability of (1) and (2), for this we need to prove that

$$Y_{i,jk}^\mu = Y_{i,kj}^\mu \quad (26)$$

and

$$N_{A,jk}^\mu = N_{A,kj}^\mu \quad (27)$$

On the other hand, if (11) and (12) are satisfied, the system (1) and (2) possesses solution $\{Y_{,i}^\mu, N_A^\mu\}$. By hypothesis we know that the Gauss-Codazzi-Ricci equations are satisfied for b_{ijA} and A_{iAB} values. Therefore:

- Take (6), using the fact that we can have $\Gamma's = 0$ on the some point, we rewrite (6). Then we have for all $Y_{,n}^\nu$ the projections on $T_p V_n$ of the vectors $Y_{,ijk}^\mu$ and $Y_{,ikj}^\mu \in T_p V_D$ are the same,

$$Y_{,ijk}^\mu Y_{,n}^\nu \eta_{\mu\nu} = Y_{,ikj}^\mu Y_{,n}^\nu \eta_{\mu\nu}, \quad (28)$$

where the expressions for $Y_{,ijk}^\mu$ and $Y_{,ikj}^\mu$ are given by (5).

- Now take (8) and using eqs (13), (14) and (15) from section (2) we have for all N_B^μ that the projections on $T_p(V_n)^\perp$ of the vectors $Y_{,ijk}^\mu$ and $Y_{,ikj}^\mu \in T_p V_D$ are equal,

$$Y_{,ijk}^\mu N_C^\nu \eta_{\mu\nu} = Y_{,ikj}^\mu N_C^\nu \eta_{\mu\nu}. \quad (29)$$

Wherein $TV_D = TV_n \oplus T(V_n)^\perp$ and the projections vectors $Y_{,ijk}^\mu$ and $Y_{,ikj}^\mu$ in the spaces $T_p V_n$ and $T_p(V_n)^\perp$ are equal, we obtain (11).

Now we prove (12). Take (10) and using eqs. (13), (14) and (15) from section (2), on the geodesic system we have for all N_B^ν , the projections on $T_p(V_n)^\perp$ of vectors $N_{A,jk}^\mu$ and $N_{A,kj}^\mu \in T_p V_D$ are equal,

$$N_{A,jk}^\mu N_B^\nu \eta_{\mu\nu} = N_{A,kj}^\mu N_B^\nu \eta_{\mu\nu}, \quad (30)$$

where the expressions for $N_{A,jk}^\mu$ and $N_{A,kj}^\mu$ are given by (9).

Observe that the projections of these vectors are equal zero,

$$N_{A,jk}^\mu Y_{,n}^\nu \eta_{\mu\nu} = 0, \quad \forall Y_{,n}^\nu \in T_p V_n \quad (31)$$

$$N_{A,kj}^\mu Y_{,n}^\nu \eta_{\mu\nu} = 0, \quad \forall Y_{,n}^\nu \in T_p V_n \quad (32)$$

therefore

$$N_{A,jk}^\mu Y_{,n}^\nu \eta_{\mu\nu} = N_{A,kj}^\mu Y_{,n}^\nu \eta_{\mu\nu}. \quad (33)$$

Wherein $TV_D = TV_n \oplus T(V_n)^\perp$ and the projections of vectors $N_{A,jk}^\mu$ and $N_{A,kj}^\mu$ in the spaces $T_p V_n$ and $T_p(V_n)^\perp$ are equal, we obtain (12). ■

4 Conclusions

In this paper we have especially investigated a mathematical support for braneworld theory with the new proof of fundamental theorem of submanifolds. Through of a mechanism of the successive immersions we always can found the ground state of the bulks, i.e., a pseudo-Euclidean bulk. We can always find a space (bulk or braneworld) which contains locally immersed another bulk. Then the theorem appears to warrant that each braneworld can be immersed in the pseudo-Euclidean bulk. Certainly the geometric entities that appear of that immersion formalism, for example, the second fundamental form and the coefficients of normal connection should be interpreted physically in agreement with the theory of braneworlds. We are preparing an article explaining physically this mechanism.

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